

ON THE GEOMETRY OF HAMILTONIAN VECTOR FIELDS

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Abstract. It is well known that geometry of vector fields play important role not only in mathematics but also it is important in many branches of science. Among vector fields Hamiltonian vector fields generate special class and they usually used to describe a motion of rigid body around fixed point.

Let M be a smooth Riemannian manifold of dimension n with the Riemannian metric g , ∇ - the Levi-Civita connection, $\langle \cdot, \cdot \rangle$ - inner product defined by the Riemannian metric g .

We denote by $V(M)$ the set of all smooth vector fields defined on M , through $[X, Y]$ Lie bracket of vector fields $X, Y \in V(M)$. The set $V(M)$ is a Lie algebra with Lie bracket.

Let's consider some set $D \subset V(M)$, which contains finite or infinite number of smooth vector fields. For a point $x \in M$ through $t \rightarrow X^t(x)$ we will denote the integral curve of a vector field X passing through a point x at $t = 0$. Map $t \rightarrow X^t(x)$ is defined in some domain $I(x) \subset \mathbb{R}$, which generally depends on field X and point x .

Definition 1.1 The orbit $L(x)$ of a system D of vector fields through a point x is the set of points y in M such that there exist $t_1, t_2, \dots, t_k \in \mathbb{R}$ and vector fields $X_1, X_2, \dots, X_k \in D$ such that

$$y = X_k^{t_k}(X_{k-1}^{t_{k-1}}(\dots(X_1^{t_1}(x))\dots)),$$

where k is an arbitrary positive integer.

There exist many papers devoted to topology and geometry of orbits

In this paper we study the geometry of orbits of Hamiltonian vector fields.

Let us recall a notion of Hamiltonian vector field. In order to define notion of Hamiltonian vector field we need to define Poisson bracket of functions.

Definition 1.2 A *Poisson bracket* on a smooth manifold M is an operation that assigns a smooth real-valued function $\{F, H\}$ on M to each pair F, H of smooth, real-valued functions, with the basic properties:

(a) *Bilinearity*:

$$\begin{aligned}\{\lambda F + \mu P, H\} &= \lambda\{F, H\} + \mu\{P, H\}, \\ \{F, \lambda H + \mu P\} &= \lambda\{F, H\} + \mu\{F, P\}, \quad \lambda, \mu \in \mathbb{R};\end{aligned}$$

(b) *Skew-Symmetry*:

$$\{F, H\} = -\{H, F\};$$

(c) *Jacobi Identity*:

$$\{\{F, H\}, P\} + \{\{P, F\}, H\} + \{\{H, P\}, F\} = 0;$$

(d) *Leibniz' Rule*:

$$\{F, H \cdot P\} = \{F, H\} \cdot P + H \cdot \{F, P\}.$$

(Here $\cdot$ denotes the ordinary multiplication of real-valued functions.)

A manifold M with a Poisson bracket is called a *Poisson manifold*, the bracket defining a Poisson structure on M . The notion of a Poisson manifold is slightly more general than that of a symplectic manifold, or manifold with Hamiltonian structure; in particular, the underlying manifold M need not be even-dimensional. This is borne out by the standard examples from classical mechanics.

Let M be the Euclidean space R^m , with coordinates

$$(p, q, z) = (p^1, \dots, p^n, q^1, \dots, q^n, z^1, \dots, z^l),$$

where $m = 2n + l$.

If $F(p, q, z)$ and $H(p, q, z)$ are smooth functions, we define their Poisson bracket to be the function:

$$\{F, H\} = \sum_{i=1}^n \left(\frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p^i} - \frac{\partial F}{\partial p^i} \frac{\partial H}{\partial q^i} \right).$$

This bracket is clearly bilinear and skew-symmetric; the verifications of the Jacobi identity and the Leibniz rule are simple exercises. We note the particular bracket identities:

$$\begin{aligned}\{p^i, p^j\} &= 0, \quad \{q^i, q^j\} = 0, \quad \{q^i, p^j\} = \delta_j^i, \\ \{p^i, z^k\} &= \{q^i, z^k\} = \{z^t, z^k\} = 0.\end{aligned}$$

in which i and j run from 1 to n , when t and k run from 1 to l . Here δ_j^i is the Kronecker symbol, which is 1 if $i = j$ and 0 otherwise.

Definition 1.3. Let M be a Poisson manifold and $H: M \rightarrow R$ a smooth function. The *Hamiltonian vector field* associated with H is the unique smooth vector field $sgrad H$ on M satisfying

$$sgrad H(F) = \{F, H\} = -\{H, F\}$$

for every smooth function $F: M \rightarrow R$.

The equations governing the flow of $sgradH$ are referred to as *Hamilton's equations* for the *Hamiltonian function* H .

In the case of the canonical Poisson bracket on R^m ($m = 2n + l$), the Hamiltonian vector field to any $H(p, q, z)$, as clearly, corresponds

$$sgradH = \sum_{i=1}^n \left(\frac{\partial H}{\partial p^i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p^i} \right)$$

The corresponding flow is obtained by integrating the system of ordinary differential equations

$$\begin{aligned} \frac{dp^i}{dt} &= -\frac{\partial H}{\partial q^i}, \quad \frac{dq^i}{dt} = \frac{\partial H}{\partial p^i}, \quad i = 1, \dots, n, \\ \frac{dz^l}{dt} &= 0, \quad l = 1, \dots, l. \end{aligned}$$

There is a fundamental connection between the Poisson bracket of two functions and the Lie bracket of their associated *Hamiltonian vector fields*, which forms the basis of much of the theory of Hamiltonian systems. It is well known following theorem :

Theorem 1. Let M be a Poisson manifold and $\{F, H\}: M \rightarrow R$ are smooth functions with corresponding Hamiltonian vector fields $sgradF$, $sgradH$. The Hamiltonian vector field associated with the Poisson bracket of F and H is, up to sign, the Lie bracket of the two Hamiltonian vector fields: $sgrad\{F, H\} = [sgradH, sgradF]$.

The Hamiltonian vector field associated with $H(x)$ has the form

$$sgradH = \sum_{i=1}^n \left(\sum_{j=1}^n \{x^i, x^j\} \frac{\partial H}{\partial x^j} \frac{\partial}{\partial x^i} \right).$$

Let $F(x)$ be a second smooth function. We obtain the basic formula

$$\{F, H\} = \sum_{i=1}^n \sum_{j=1}^n \{x^i, x^j\} \frac{\partial F}{\partial x^i} \frac{\partial H}{\partial x^j}$$

for the Poisson bracket.

This basic brackets $A^{ij}(x) = \{x^i, x^j\}$ $i, j = 1, \dots, m$ are called the *structure functions* of the Poisson manifold M with respect to the given local coordinates.

A skew-symmetric $m \times m$ matrix $A(x) = \{A^{ij}(x)\}$ called the *structure matrix* of M . If the Poisson bracket is non-degenerate ($\det(A^{ij}) \neq 0$ everywhere on M), then the Poisson manifold is called a *symplectic manifold*. The symplectic structure in this case has the form $\omega = A_{ij} dx^i \wedge dx^j$ where A_{ij} are components of the matrix inverse to (A^{ij}) .

Let us consider functions $H_1, H_2: \mathbb{R}^4 \rightarrow \mathbb{R}$ on the Euclidean four dimensional space $\mathbb{R}^4$ with cartezian coordinates p_1, p_2, q_1, q_2 which are given by the formulas

$$H_1 = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2)$$

$$H_2 = \frac{1}{2}(p_1^2 + p_2^2 - q_1^2 + q_2^2).$$

The Hamiltonian vector fields corresponding to H_1 and H_2 have following forms:

$$sgradH_1 = -q_1 \frac{\partial}{\partial p^1} - q_2 \frac{\partial}{\partial p^2} + p_1 \frac{\partial}{\partial q^1} + p_2 \frac{\partial}{\partial q^2} \quad (1)$$

$$sgradH_2 = q_1 \frac{\partial}{\partial p^1} - q_2 \frac{\partial}{\partial p^2} + p_1 \frac{\partial}{\partial q^1} - p_2 \frac{\partial}{\partial q^2}$$

Theorem 2. The orbits of vector fields (1) generate singular foliation regular leaf of which is three dimensional surface with zero Gauss curvature.

Following sets are singular leaves:

$L_1 = \{0,0,0,0\}$ – zero dimensional orbit;

$L_2 = \{(0, p_2, 0, q_2): p_2^2 + q_2^2 = c, c > 0\}$ – one dimensional orbit.

We also have following leaf: $L_3 = \{(p_1, 0, q_1, 0): p_1^2 + q_1^2 > 0\}$ – two dimensional plane (p_1, q_1) without the point $(0,0)$.

Now let us consider functions

$$H_1 = p_1 q_1 + p_2 q_2$$

$$H_2 = p_1 q_2 - p_2 q_1$$

and hamiltonian vector fields

$$sgradH_1 = -p_1 \frac{\partial}{\partial p^1} - p_2 \frac{\partial}{\partial p^2} + p_1 \frac{\partial}{\partial q^1} + p_2 \frac{\partial}{\partial q^2} \quad (2)$$

$$sgradH_2 = -p_2 \frac{\partial}{\partial p^1} + p_1 \frac{\partial}{\partial p^2} + q_2 \frac{\partial}{\partial q^1} - q_1 \frac{\partial}{\partial q^2}$$

Theorem 3. The orbits of vector fields (2) generate singular foliation regular leaf of which is two dimensional surface zero Gauss curvature and zero Gauss torsion.

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